APS/123-QED

Berry Phase and Adiabatic Breakdown in Optical Modulator

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(Dated: February 1, 2008)

We consider an *all in-fiber* optical modulator based on a ring resonator configuration. The case of adiabatic to nonadiabatic transition is considered, where the geometrical (Berry) phase acquired in a round trip along the ring changes abruptly by π. Degradation of the responsivity of the modulator due to finite linewidth of the optical input is discussed. We show that the responsivity of the proposed modulator can be significantly enhanced with optimum design and compare with other configurations.

PACS numbers: 42.79.Hp, 42.60.Da, 42.81.i

I. INTRODUCTION

Optical modulators are devices of great importance for optical communication and other fields. In these devices some external perturbation, e.g. electric or magnetic fields, is employed to modulate the transmission $\mathcal{T}$ between the input and output optical ports. One of the key property of an optical modulator is the responsivity, namely the dependence of $\mathcal{T}$ on the applied perturbation. Enhancing the responsivity is highly desirable in many applications. As is shown in Ref. [1], the linearity of optical modulators imposes in general an upper bound on their responsivity. One way of achieving high responsivity is by employing a resonator configuration with high quality factor Q. The multiple back and forth reflections occurring in a resonator allow enhancing the responsivity in comparison with the case of reflectionless optical path. Such a ring resonator was considered recently by Yariv [2], [3] and implemented experimentally [4], [5]. It was shown that high enhancement is achieved when critical coupling occurs, namely when the power entering the resonator from the input port equals the output dissipation power. On the other hand, one of the drawbacks of a resonator configuration is the limited optical bandwidth. In some cases finite linewidth of the optical input $\Delta\omega$ may lead to broadening of the resonance and thus reducing the responsivity. Such broadening can be avoided only when $\Delta\omega/\omega << \lambda/QL$, where λ is the wavelength and L is a characteristic length of the resonator.

In this paper we consider a ring resonator similar to the one discussed in [2], [3]. However while Ref. [2], [3] considered the case of polarization independent evolution, here we study the case of finite birefringence $\boldsymbol{\kappa}(s)$ along the optical path (s is a coordinate along the optical path). We first consider the case of adiabatic evolution, when $\boldsymbol{\kappa}$ changes slowly. In this case it is convenient to express the state of polarization (SOP) in the basis of local eigenvectors. In this basis the equations of motion of both polarization amplitudes can be decoupled to the lowest order in the adiabatic expansion. Next we consider the case of adiabatic breakdown, namely the transition into the regime where the adiabatic approximation does not hold. In this case the geometrical (Berry) [6] phase acquired in a round trip along the ring changes abruptly by π. We show that this abrupt change can be employed for achieving high responsivity. Note that similar adiabatic breakdown was considered in Ref. [7] for the case of spin 1/2 electrons in coherent mesoscopic conductors with spin-orbit interaction (see also Ref. [8]).

Such an optical modulator based on adiabatic breakdown can be implemented in a variety of different configurations. Here we demonstrate these effects by considering a relatively simple example of a modulator based on a fiber ring resonator having both intrinsic and externally applied birefringence. The intrinsic birefringence along the ring in our example is linear. As we discuss below, it can be induced using a standard polarization maintaining fiber being twisting and tapered to realized the desired birefringence. The externally applied birefringence used for modulation is based in our example on magneto-optic effect [9], [10]. This effect allows inducing circular birefringence in the fiber, being proportional to the Verdet constant characterizing the material and to the component of the applied magnetic field along the direction of propagation. We employ both analytical and numerical calculations to study the responsivity of the system. We find enhanced responsivity when operating in the adiabatic breakdown regime.

II. FIBER RING RESONATOR

Consider a fiber ring resonator as seen in Fig. 1. It consists of a fiber ring coupled to input and output ports using a directional coupler.

The SOP at each point along the fiber is described as a *spinor* with two components associated with the amplitudes of two orthonormal polarization states. As is discussed in appendix A and appendix B, we use the local eigenvectors as a basis to express the SOP. The associated amplitudes are $E_{\uparrow}$ and $E_{\downarrow}$ respectively. The directional coupler is assumed to have coupling constants independent of the SOP. Moreover, the coupling is as-

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FIG. 1: The fiber ring resonator.

sumed lossless, thus the coupling matrix is unitary

$$\begin{pmatrix} E^{b_1}_{\uparrow} \\ E^{b_2}_{\uparrow} \\ E^{b_1}_{\downarrow} \\ E^{b_2}_{\downarrow} \end{pmatrix} = \begin{pmatrix} t & r & 0 & 0 \\ -r^* & t^* & 0 & 0 \\ 0 & 0 & t & r \\ 0 & 0 & -r^* & t^* \end{pmatrix} \begin{pmatrix} E^{a_1}_{\uparrow} \\ E^{a_2}_{\uparrow} \\ E^{a_1}_{\downarrow} \\ E^{a_2}_{\downarrow} \end{pmatrix}, \tag{1}$$

where

$$|t|^2 + |r|^2 = 1. \tag{2}$$

Integrating the equation of motion along the ring leads in general to a linear relation between the amplitudes at both ends

$$\begin{pmatrix} E^{a_2}_{\uparrow} \\ E^{a_2}_{\downarrow} \end{pmatrix} = \hat{M} \begin{pmatrix} E^{b_2}_{\uparrow} \\ E^{b_2}_{\downarrow} \end{pmatrix}, \tag{3}$$

where

$$\hat{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{4}$$

Using 1, 3, and 2 one can find a linear relation between the amplitudes in the input and output ports of the modulator

$$\begin{pmatrix} E^{b_1}_{\uparrow} \\ E^{b_1}_{\downarrow} \end{pmatrix} = \hat{S} \begin{pmatrix} E^{a_1}_{\uparrow} \\ E^{a_1}_{\downarrow} \end{pmatrix}, \tag{5}$$

where the matrix $\hat{S}$ is given by

$$\hat{S} = \frac{1 - t\hat{M}^{-1}}{t^* - \hat{M}^{-1}}. \tag{6}$$

Note that if $\hat{M}$ is unitary (namely, $\hat{M}^{-1} = \hat{M}^{\dagger}$) and 2 holds then, as expected, $\hat{S}$ is unitary as well. Note also that if $\hat{M}$ is diagonal (namely, $M_{12} = M_{21} = 0$) the following holds [2], [3]

$$\hat{S} = \begin{pmatrix} \frac{t - M_{11}}{1 - M_{11} t^*} & 0 \\ 0 & \frac{t - M_{22}}{1 - M_{22} t^*} \end{pmatrix}. \tag{7}$$

To find the matrix $\hat{M}$ one has to integrate the equation of motion A9 along the close curve defined by the ring. In the adiabatic limit, to be discussed in the next section, the solution can be found analytically. In the following section the case of adiabatic breakdown is discussed, where both analytical approximations and numerical calculations are employed to integrate the equation of motion A9.

III. THE ADIABATIC CASE

In the case where the adiabatic approximation can be applied the matrix $\hat{M}$ is given by

$$\hat{M} = \begin{pmatrix} \exp(i\delta_{\uparrow}) & 0 \\ 0 & \exp(i\delta_{\downarrow}) \end{pmatrix}, \tag{8}$$

where $\delta_{\uparrow}$ and $\delta_{\downarrow}$ are given by equations B23 and B24 respectively.

In the more general case the ring may have internal loss. Assuming the loss is polarization independent, one has

$$\hat{M} = (1 - \xi_l) \begin{pmatrix} \exp(i\delta_{\uparrow}) & 0 \\ 0 & \exp(i\delta_{\downarrow}) \end{pmatrix}, \tag{9}$$

where $0 \leq \xi_l \leq 1$ is real. Thus using 7

$$\frac{E^{b_1}_{\sigma}}{E^{a_1}_{\sigma}} = \frac{t - (1 - \xi_l)\exp(i\delta_{\sigma})}{1 - (1 - \xi_l)\exp(i\delta_{\sigma})\, t^*}, \tag{10}$$

where $\sigma \in \{\uparrow, \downarrow\}$. Using the notation $t = (1 - \xi_c)\exp(i\theta_t)$, where $0 \leq \xi_c \leq 1$ is real, and $\vartheta = \delta_{\sigma} - \theta_t$ one gets

$$\frac{E^{b_1}_{\sigma}}{E^{a_1}_{\sigma}} = \exp(i\theta_t) \frac{1 - \xi_c - (1 - \xi_l)\exp(i\vartheta)}{1 - (1 - \xi_l)(1 - \xi_c)\exp(i\vartheta)}. \tag{11}$$

Near resonance $\vartheta << 1$. Moreover, assuming $\xi_l << 1$ and $\xi_c << 1$, one finds

$$\frac{E^{b_1}_{\sigma}}{E^{a_1}_{\sigma}} \simeq \exp(i\theta_t) \frac{\xi_l - \xi_c - i\vartheta}{\xi_l + \xi_c - i\vartheta}. \tag{12}$$

Critical coupling occurs when $\xi_l = \xi_c \equiv \xi$. In this case the transmission amplitude $E_\sigma^{b_1}/E_\sigma^{a_1}$ vanishes at resonance. The transmission probability in this case is given by

$$\mathcal{T}(\vartheta) \simeq \frac{(Q\vartheta)^2}{1+(Q\vartheta)^2}, \tag{13}$$

where $Q = 1/2\xi$. Thus, high responsivity can be achieved when operating close to a resonance with high Q factor.

IV. BROADENING DUE TO FINITE LINEWIDTH

As was discussed in the previous section, relatively high responsivity can be achieved when operating close to a resonance. However, as we discuss below, the price one has to pay for that is limited bandwidth.

Consider the case where the optical input has some finite linewidth $\Delta\omega$. As a result the phase factor ϑ will acquire a linewidth given by

$$\Delta\vartheta = 2\pi\frac{\Delta\omega}{\omega}\frac{L}{\lambda}. \tag{14}$$

Consider the case of a polychromatic optical input and assume that the probability distribution of ϑ is Lorenzian with a characteristic width $\Delta\vartheta$

$$f(\vartheta') = \frac{1}{\pi\Delta\vartheta}\frac{1}{1+\left(\frac{\vartheta'-\vartheta}{\Delta\vartheta}\right)^2}. \tag{15}$$

Averaging using this distribution and Eq. 13, and employing the residue theorem for evaluating the integral one finds

$$\begin{aligned}\bar{\mathcal{T}}(\vartheta) &= \int_{-\infty}^{\infty} d\vartheta' f(\vartheta')\,\mathcal{T}(\vartheta') \\ &= 1 - \frac{1}{1+Q\Delta\vartheta}\frac{1}{1+\left(\frac{Q\vartheta}{1+Q\Delta\vartheta}\right)^2}.\end{aligned} \tag{16}$$

Thus, for this case broadening can be avoided only if $Q\Delta\vartheta << 1$ or $\Delta\omega/\omega << \lambda/QL$.

V. ADIABATIC BREAKDOWN

While in the previous case both adiabatic SOP are effectively decoupled, we consider now the transition between adiabatic and non-adiabatic regimes.

The birefringence along the fiber ring is described by the vector $\boldsymbol{\kappa}(s)$ (see appendix A). Consider the case where in some section of the ring $\boldsymbol{\kappa}(s)$ is close to the degeneracy point at the origin $\boldsymbol{\kappa} = 0$. In this case small perturbation applied to $\boldsymbol{\kappa}(s)$ can result in a large change in the geometrical phase B20 and B21. This can be seen by considering, for example, the case of a planar curve $\boldsymbol{\kappa}(s)$. In this case the solid angle is given by $\Omega = 2\pi n$, where n is the winding number of the curve $\boldsymbol{\kappa}(s)$ around the origin. As the curve $\boldsymbol{\kappa}(s)$ crosses the origin at some point, n changes abruptly by one, leading thus to an abrupt change in the geometrical phase. Note however that near this transition when $|\boldsymbol{\kappa}(s)|$ is small the adiabatic approximation breaks down and alternative approaches are needed.

As an example for such a transition we consider a ring resonator for which the close curve $\boldsymbol{\kappa}(s)$ has the shape seen in Fig. 2 (c) in the unperturbed case. This curve is made of 'half circle' section in the 1-3 plane (the linear birefringence plane) and a 'diameter' section along the κ_3 axis crossing the origin. Such a structure can be realized by using a polarization maintaining fiber and by employing fiber tapering techniques. The half circle section can be made out of a Möbius like ring made of the polarization maintaining fiber. After welding the two ends of the twisted fiber to form the Möbius structure one can employ tapering techniques to form the 'diameter' section.

The curve $\boldsymbol{\kappa}(s)$ is perturbed by applying a magnetic field on part of the 'diameter' section of the fiber ring. Such a perturbation contributes circular birefringence in the κ_2 direction (see Fig. 2 (a) and (e). The relatively high value of the Verdet constant in common optical fibers allows significant magneto-optic effect with moderate applied magnetic fields. While the adiabatic approximation totally breaks down in the unperturbed case of Fig. 2 (c) when the curve $\boldsymbol{\kappa}(s)$ crosses the origin, the perturbation transforms the system into the regime where adiabaticity holds. As is shown below, the responsivity of the system is relatively high when operating near this transition between the adiabatic and non-adiabatic regimes.

The 'half circle' section is analyzed in appendix C. As can be seen in Fig. 6, the Zener transition probability p_z vanishes for a series of points denoted as Λ_n. In our example we chose Λ to be the first zero of $p_z(\Lambda)$, namely $\Lambda = \Lambda_1 = 1.022$. One advantage of choosing one of the zeros of $p_z(\Lambda)$, where p_z obtains a local minimum, is the fact that p_z is only weakly affected by small deviations of $\boldsymbol{\kappa}(s)$ from the ideal 'half circle' curve. For the parameter γ we chose the value $\gamma = 1$. As can be seen from Fig. 2 (d) for this choice the evolution along the 'half circle' section transform the polarization vector on the Bloch sphere from the pole on the negative P_z axis to the opposite pole on the positive P_z axis. The fiber length of this section is $2\Lambda_1/\gamma$.

The rest of the fiber ring has a birefringence given by $\boldsymbol{\kappa}(s) = \boldsymbol{\kappa}_0(s) + \boldsymbol{\kappa}_1(s)$, where $\boldsymbol{\kappa}_0(s)$ is the unperturbed birefringence forming the 'diameter' section and $\boldsymbol{\kappa}_1(s)$ is the perturbation induced by the magnetic field. The

FIG. 2: The birefringence $\boldsymbol{\kappa}(s)$ and polarization $\mathbf{P}(s)$ along the fiber ring. Plots (c) and (d) shows the unperturbed case, while in (a) and (b) the perturbation parameter is $\alpha = -0.2$, and in (e) and (f) $\alpha = 0.2$.

unperturbed part is assumed to be given by

$$\boldsymbol{\kappa}_0(s) = -\left(0, 0, \frac{\Lambda_1 \gamma^2}{\beta} s\right), \tag{17}$$

where $|s| < \beta/\gamma$. In our numerical example the dimensionless parameter β is given the value $\beta = 5$. The perturbation due to the applied magnetic field gives rise to birefringence given by

$$\boldsymbol{\kappa}_1(s) = \left(0, \frac{\alpha}{1 + \exp A\left[\left(\frac{\gamma s}{\beta}\right)^2 - B^2\right]}, 0\right), \tag{18}$$

where $A = 50$ and $B = 0.6$ in our numerical example. Thus, the magnetic field is applied to a fiber section of length $2B\beta/\gamma$ and drops down to zero abruptly outside this section (due to the large value chosen for the parameter A). The coupling constants in the numerical example are $\xi_c = 10^{-2}$ and $\xi_l = 10^{-4}$.

The equation of motion along the fiber ring is integrated numerically as described in appendix A. This allows calculating the evolution of the polarization vector on the Bloch sphere (see Fig. 2 (b), (d), and (f)). The same calculation yields also the matrix $\hat{M}$. The off-diagonal matrix elements allow calculating the Zener transition probability $|M_{12}|^2 = |M_{21}|^2$ (see Fig. 3 (a) solid line). The curve shows the gradual transition between the non-adiabatic limit where $|\alpha| << 1$ and the adiabatic limit $|\alpha| >> 1$. An approximated analytical expression for the Zener probability in a similar case where the curve $\boldsymbol{\kappa}(s)$ is an infinite straight line was derived in appendix C. The result in Eq. C25 can be used to estimate approximately the Zener transition probability for the present example

$$p_z = \exp\left(-\frac{\pi\beta\alpha^2}{\Lambda_1\gamma^2}\right). \tag{19}$$

The estimate in Eq. 19 is shown in Fig. 3 (a) as a dashed line. The deviation between the numerical and analytical results is originated mainly by the fact that the straight line section in $\boldsymbol{\kappa}(s)$ is finite while the analytical analysis assumes an infinite straight line. Moreover, the analytical result is expected to hold only in the limit where $|p_z| << 1$ as it is evaluated only to lowest order in the adiabatic expansion.

Figure 3 (b) shows the phase of both diagonal matrix elements of $\hat{M}$. In both cases the phase changes abruptly by π near $\alpha = 0$. This is originated by the sharp change of the solid angle Ω by 2π near $\alpha = 0$ (see Eq. B23 and B24). The optical modulator discussed in the present work employs this sharp change to achieve high responsivity.

Figure 3 (c) shows the transmission probability into both SOP, $P_{11} = |S_{11}|^2$ (solid line) and $P_{21} = |S_{21}|^2$ (dashed line) of the entire modulator. For both cases, the full width half maximum (FWHM) is $\Delta\alpha = 5.1 \times 10^{-3}$.

VI. DISCUSSION

As we have seen, the ring resonator can serve as an optical modulator with high responsivity when operated near one of its resonances. Two regimes of operation were considered, the adiabatic one, and the non-adiabatic one. In what follows we compare between both regimes by considering the following points.

Optical Source Linewidth - In the adiabatic limit, when the equations of motion in the adiabatic basis become decoupled, the only effect of the external perturbation is on the phases acquired along the fiber ring.

FIG. 3: Dependence on the perturbation amplitude α. (a) Zener transition probability, calculated numerically (solid line) and estimated using Eq. C25. (b) Phase of M_{11} (solid line) and of M_{22} (dashed line). (c) transmission probability into both SOP, $P_{11} = |S_{11}|^2$ (solid line) and $P_{21} = |S_{21}|^2$ (dashed line).

The dependence of the dynamical phase on wavelength gives rise to broadening of resonances when operating with an optical input having finite linewidth. In the general nonadiabatic regime, however, the external perturbation can affect not only the phase factors but also the SOP as it evolves along the close fiber ring. The later, being wavelength independent gives rise to a modified dependence on the optical source linewidth.

Critical Coupling - In the adiabatic regime full modulation between zero and one of the transmission probability $\mathcal{T}$ is possible only when critical coupling occurs, namely $\xi_c = \xi_l$, (see Eq. 12). In practice, fulfilling this condition when $\xi_c = \xi_l << 1$ is difficult. However, this condition is not essential in the general non-adiabatic case. As can be seen in Fig. 3 (c) full modulation is achieved, even thought for this example $\xi_c = 100\xi_l$.

Responsivity - The responsivity of the ring resonator device can be characterized by the FWHM and height of the resonance near which the device is being operated. As was discussed above, $\Delta\alpha = 5.1 \times 10^{-3}$ for the example presented in Fig. 3. For the same parameters the FWHM of the resonances in the adiabatic regime $|\alpha| >> 1$ can be evaluated using Eq. 12 yielding $\Delta\alpha = 1.7 \times 10^{-3}$. However, as was discussed above, since the coupling is not critical, the modulation is not full in the adiabatic case. Note that in general the responsivity has an upper bound imposed by the linearity of the system [1]. It can be shown that for both cases, the obtained responsivity is of the same order as the upper bound. A future publication will discuss this point in more details.

VII. SUMMARY

In the present work we study an optical modulator based on a fiber ring resonator. Both adiabatic and non-adiabatic regimes of operation are considered. We find that operating close to the point where the geometrical phase changes abruptly by π can allow relatively high responsivity, even when coupling is not set to be critical. Our example deals with a particular configuration of optical fiber ring with both intrinsic and externally applied birefringence. However, the same ideas can be implemented with other optical waveguides and other birefringence mechanisms.

VIII. ACKNOWLEDGEMENTS

The author would especially like to thank Avishai Eyal for many helpful conversations and invaluable suggestions. Also a discussion with Steve Lipson is gratefully acknowledged.

APPENDIX A: SOP EVOLUTION ALONG A FIBER

Consider an optical fiber winded in some spacial curve in space. Let $\mathbf{r}(s)$ be an arc-length parametrization of this curve, namely the tangent $\hat{\mathbf{s}} = d\mathbf{r}/ds$ is a unit vector. The normal unit vector $\hat{\boldsymbol{\nu}}$ and the curvature κ are defined as $d\hat{\mathbf{s}}/ds = \kappa\hat{\boldsymbol{\nu}}$. One can easily show that $\hat{\boldsymbol{\nu}} \cdot \hat{\mathbf{s}} = 0$ by taking the derivative of $\hat{\mathbf{s}} \cdot \hat{\mathbf{s}} = 1$ with respect to s. The vectors $\hat{\mathbf{s}}$, $\hat{\boldsymbol{\nu}}$ and the binormal unit vector, defined as $\hat{\mathbf{b}} = \hat{\mathbf{s}} \times \hat{\boldsymbol{\nu}}$, form a local triplet orthonormal coordinate frame known as Serret - Frenet frame [11], [12] (see Fig. 4). By taking the derivative of $\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\nu}} = 0$ with respect to s one finds $\hat{\mathbf{s}} \cdot d\hat{\boldsymbol{\nu}}/ds = -\kappa$. Similarly, by taking the derivative of $\hat{\mathbf{b}} \cdot \hat{\boldsymbol{\nu}} = 0$ with respect to s one finds $\hat{\mathbf{b}} \cdot d\hat{\boldsymbol{\nu}}/ds = -\hat{\boldsymbol{\nu}} \cdot d\hat{\mathbf{b}}/ds$. Using the definition $\hat{\mathbf{b}} = \hat{\mathbf{s}} \times \hat{\boldsymbol{\nu}}$ one finds $d\hat{\mathbf{b}}/ds = \hat{\mathbf{s}} \times d\hat{\boldsymbol{\nu}}/ds$. Thus $\hat{\mathbf{s}} \cdot d\hat{\mathbf{b}}/ds = 0$. Moreover, by taking the derivative of $\hat{\mathbf{b}} \cdot \hat{\mathbf{b}} = 1$ with respect to s on finds $\hat{\mathbf{b}} \cdot d\hat{\mathbf{b}}/ds = 0$. Thus $d\hat{\mathbf{b}}/ds$ is parallel to $\hat{\boldsymbol{\nu}}$. The torsion τ is defined as $d\hat{\mathbf{b}}/ds = -\tau\hat{\boldsymbol{\nu}}$. The above definitions and relations can be summarized as follows

$$\frac{d}{ds}\begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\nu}} \\ \hat{\mathbf{b}} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{s}} \\ \hat{\boldsymbol{\nu}} \\ \hat{\mathbf{b}} \end{pmatrix} . \qquad \text{(A1)}$$

The equation of motion along the optical ray defined by the fiber can be obtained using the transport equation of geometric optics [13] for the electric field phasor $\mathbf{E}_0$

$$\begin{aligned} 2(\nabla\psi \cdot \nabla)\mathbf{E}_0 + \mathbf{E}_0\left[\nabla^2\psi - \nabla(\ln\mu) \cdot \nabla\psi\right] \qquad \text{(A2)} \\ +2\left[\mathbf{E}_0 \cdot \nabla(\ln n)\right]\nabla\psi = 0, \end{aligned}$$

FIG. 4: The Serret-Frenet frame.

where ψ is the eikonal, n is the index of refraction, and μ is the permeability. Define the unit vector $\hat{\mathbf{e}}_0 = \mathbf{E}_0/\sqrt{\mathbf{E}_0 \cdot \mathbf{E}_0^*}$ in the direction of $\mathbf{E}_0$. In terms of $\hat{\mathbf{e}}_0$ the transport equation reads

$$\frac{d}{ds}\hat{\mathbf{e}}_0 = -\kappa\left(\hat{\mathbf{e}}_0 \cdot \hat{\boldsymbol{\nu}}\right)\hat{\mathbf{s}}. \tag{A3}$$

Expressing the unit vector $\hat{\mathbf{e}}_0$ in the Serret - Frenet frame

$$\hat{\mathbf{e}}_0 = e_\nu \hat{\boldsymbol{\nu}} + e_b \hat{\mathbf{b}}, \tag{A4}$$

one finds using A1

$$\frac{de_\nu}{ds}\hat{\boldsymbol{\nu}} + \frac{de_b}{ds}\hat{\mathbf{b}} + e_\nu\left(-\kappa\hat{\mathbf{s}} + \tau\hat{\mathbf{b}}\right) - e_b\tau\hat{\boldsymbol{\nu}} = -\kappa e_\nu \hat{\mathbf{s}}, \tag{A5}$$

thus, using the Dirac *ket* notation

$$|e\rangle \doteq \begin{pmatrix} e_\nu \\ e_b \end{pmatrix}, \tag{A6}$$

one finds

$$\frac{d}{ds}|e\rangle = i\mathcal{K}_g |e\rangle\,, \tag{A7}$$

where the geometrical birefringence $\mathcal{K}_g$ is given by

$$\mathcal{K}_g = \tau \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{A8}$$

Equation A7 is known as Rytov's law [13]. In the more general case where other birefringence mechanisms are present the equation of motion reads

$$\frac{d}{ds}|e\rangle = i\mathcal{K}|e\rangle\,, \tag{A9}$$

where $\mathcal{K} = \mathcal{K}_g + \mathcal{K}_f$, and $\mathcal{K}_f$ is the birefringence in the fiber due to intrinsic structure or due to elasto-optic or electro-optic of magneto-optic effects.

In a lossless fiber the matrix $\mathcal{K}$ is Hermitian. For this case it is convenient to expresses $\mathcal{K}$ as

$$\mathcal{K} = k_0 I + \boldsymbol{\kappa} \cdot \boldsymbol{\sigma}, \tag{A10}$$

where I is the 2 by 2 identity matrix, k_0 is a real scalar, $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \kappa_3)$ is a three-dimensional real vector and the components of the Pauli matrix vector $\boldsymbol{\sigma}$ are given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A11}$$

The s - evolution operator $u(s, s_0)$ of the equation of motion A9 relates an initial state $|e(s_0)\rangle$ with a final state at some $s > s_0$

$$|e(s)\rangle = u(s, s_0)\,|e(s_0)\rangle\,. \tag{A12}$$

It can be expressed as

$$u(s, s_0) = \lim_{N\to\infty} \prod_{n=1}^{N} \exp\left[i\frac{\Delta s}{N}\mathcal{K}(s_n)\right], \tag{A13}$$

where $\Delta s = s - s_0$, and $s_n = s_0 + n\Delta s/N$. For a finite N the above expression can be used as a numerical approximation of $u(s, s_0)$. For calculating the exponential terms in A13 it is useful to employ the following identity

$$\exp(ix\mathcal{K}) = \exp(ik_0 x)\left[I\cos(\alpha x) + i\hat{\boldsymbol{\kappa}} \cdot \boldsymbol{\sigma}\sin(\alpha x)\right], \tag{A14}$$

where the notation of Eq. A10 is being used, and $\boldsymbol{\kappa} = \hat{\boldsymbol{\kappa}}\alpha$ where $\hat{\boldsymbol{\kappa}}$ is a unit vector and $\alpha = |\boldsymbol{\kappa}|$.

The normalized SOP $|e\rangle$ can be represented as a point on the Bloch sphere indicating the expectation value of the Pauli spin vector matrix, namely

$$\mathbf{P} = \langle e|\,\boldsymbol{\sigma}\,|e\rangle\,. \tag{A15}$$

APPENDIX B: THE ADIABATIC CASE

To establish notation we review below the main results of Ref. [6]. Consider the differential equation

$$\frac{d}{ds}|\psi\rangle = i\mathcal{K}|\psi\rangle\,, \tag{B1}$$

where $|\psi\rangle$ represents N dimensional column vector and $\mathcal{K}=\mathcal{K}(s)$ is $N\times N$ Hermitian matrix. For any given value of s the Hermitian matrix $\mathcal{K}(s)$ has a set of orthonormal eigenvectors

$$\mathcal{K}\left|n\left(s\right)\right\rangle = K_n\left(s\right)\left|n\left(s\right)\right\rangle , \tag{B2}$$

where $n=1,2,...,N$ and

$$\left\langle n\left(s\right)\right|m\left(s\right)\rangle = \delta_{nm}. \tag{B3}$$

The solution can be expanded as follows

$$|\psi\rangle = \sum_n a_n\left(s\right)\exp\left[i\int_0^s ds' K_n\left(s'\right)\right]\left|n\left(s\right)\right\rangle . \tag{B4}$$

Substituting in B1 yields

$$\dot{a}_m\left(s\right) = -\sum_n a_n\left(s\right)e^{i\int_0^s ds'\left[K_n\left(s'\right)-K_m\left(s'\right)\right]}\left\langle m\left(s\right)\right|\dot{n}\left(s\right)\rangle , \tag{B5}$$

where upper-dot represents derivative with respect to s. The off-diagonal terms, given by

$$\left\langle m\left(s\right)\right|\dot{n}\left(s\right)\rangle = \frac{\left\langle m\left(s\right)\right|\dot{\mathcal{K}}\left|n\left(s\right)\right\rangle}{K_n\left(s\right)-K_m}, \tag{B6}$$

where $m\neq n$, are neglected in the adiabatic approximation. The resulting decoupled set of equations are easily solved

$$a_m\left(s\right) = a_m\left(0\right)\exp\left(i\gamma_m\right). \tag{B7}$$

where the real phase γ_m is given by

$$\gamma_m = i\int_0^s ds'\left\langle m\left(s'\right)\right|\dot{m}\left(s'\right)\rangle . \tag{B8}$$

Consider now the two dimensional case $N=2$. Using the notation of Eq. A10 and the notation $\boldsymbol{\kappa}=\hat{\boldsymbol{\kappa}}\alpha$, where $\hat{\boldsymbol{\kappa}}$ is a unit vector, given in spherical coordinates by

$$\hat{\boldsymbol{\kappa}} = \left(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta\right), \tag{B9}$$

one finds

$$\mathcal{K} = k_0 I + \alpha\begin{pmatrix}\cos\theta & \sin\theta\exp\left(-i\varphi\right)\\ \sin\theta\exp\left(i\varphi\right) & -\cos\theta\end{pmatrix}. \tag{B10}$$

The orthonormal eigenvectors are chosen to be

$$|\uparrow\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\exp\left(-\frac{i\varphi}{2}\right)\\ \sin\frac{\theta}{2}\exp\left(\frac{i\varphi}{2}\right)\end{pmatrix},\ |\downarrow\rangle = \begin{pmatrix}-\sin\frac{\theta}{2}\exp\left(-\frac{i\varphi}{2}\right)\\ \cos\frac{\theta}{2}\exp\left(\frac{i\varphi}{2}\right)\end{pmatrix}, \tag{B11}$$

and the following holds $\langle\uparrow\mid\uparrow\rangle=\langle\downarrow\mid\downarrow\rangle=1$, $\langle\uparrow\mid\downarrow\rangle=0$, and

$$\mathcal{K}\left|\uparrow\right\rangle = \left(k_0+\alpha\right)\left|\uparrow\right\rangle , \tag{B12}$$

$$\mathcal{K}\left|\downarrow\right\rangle = \left(k_0-\alpha\right)\left|\downarrow\right\rangle . \tag{B13}$$

The eigenstates $|n\rangle$ (where $n\in\{\uparrow,\downarrow\}$) are independent of k_0, thus:

$$\gamma_m = i\int_{s_1}^{s_2} ds\left\langle m\left(s\right)\right|\dot{m}\left(s\right)\rangle = i\int_{\boldsymbol{\kappa}_1}^{\boldsymbol{\kappa}_2} d\boldsymbol{\kappa}\cdot\left\langle m\left(\boldsymbol{\kappa}\right)\right|\nabla_{\boldsymbol{\kappa}}\left|m\left(\boldsymbol{\kappa}\right)\right\rangle . \tag{B14}$$

Using the expression for a gradient in spherical coordinates one finds

$$\left\langle\uparrow\right|\nabla_{\boldsymbol{\kappa}}\left|\uparrow\right\rangle = -\frac{i\hat{\boldsymbol{\varphi}}}{2\alpha}\operatorname{ctg}\theta, \tag{B15}$$

$$\left\langle\downarrow\right|\nabla_{\boldsymbol{\kappa}}\left|\downarrow\right\rangle = \frac{i\hat{\boldsymbol{\varphi}}}{2\alpha}\operatorname{ctg}\theta, \tag{B16}$$

For the case of a close path, Stock's theorem can be used to express the integral in terms of a surface integral over the surface bounded by the close curve $\boldsymbol{\kappa}(s)$

$$\gamma_m = i\oint d\boldsymbol{\kappa}\cdot\left\langle m\right|\nabla_{\boldsymbol{\kappa}}\left|m\right\rangle = i\int_S d\mathbf{a}\cdot\left(\nabla\times\left\langle m\right|\nabla_{\boldsymbol{\kappa}}\left|m\right\rangle\right). \tag{B17}$$

Expressing the curl operator in spherical coordinates one finds

$$\nabla\times\left\langle\uparrow\right|\nabla_{\boldsymbol{\kappa}}\left|\uparrow\right\rangle = \frac{i}{2}\frac{\boldsymbol{\kappa}}{\left|\boldsymbol{\kappa}\right|^3}, \tag{B18}$$

$$\nabla\times\left\langle\downarrow\right|\nabla_{\boldsymbol{\kappa}}\left|\downarrow\right\rangle = -\frac{i}{2}\frac{\boldsymbol{\kappa}}{\left|\boldsymbol{\kappa}\right|^3}, \tag{B19}$$

and

$$\gamma_\uparrow = -\frac{1}{2}\int_S d\mathbf{a}\cdot\frac{\boldsymbol{\kappa}}{\left|\boldsymbol{\kappa}\right|^3} = -\frac{1}{2}\Omega, \tag{B20}$$

$$\gamma_\downarrow = \frac{1}{2}\int_S d\mathbf{a}\cdot\frac{\boldsymbol{\kappa}}{\left|\boldsymbol{\kappa}\right|^3} = \frac{1}{2}\Omega, \tag{B21}$$

where Ω is the solid angle subtended by the close path $\boldsymbol{\kappa}(s)$ as seen from the origin. Due to the geometrical

nature of the last result, the phase factors $\gamma_\uparrow$ and $\gamma_\downarrow$ are called geometrical phases.

Thus

$$|\psi(s)\rangle = \begin{pmatrix} \exp(i\delta_\uparrow) & 0 \\ 0 & \exp(i\delta_\downarrow) \end{pmatrix} |\psi(0)\rangle, \tag{B22}$$

where

$$\delta_\uparrow = -\frac{\Omega}{2} + \int_0^s ds' \left[k_0(s') + \alpha(s')\right], \tag{B23}$$

$$\delta_\downarrow = \frac{\Omega}{2} + \int_0^s ds' \left[k_0(s') - \alpha(s')\right], \tag{B24}$$

APPENDIX C: ZENER TRANSITIONS

The set of equations B5 for the two dimensional case $N = 2$ can be written in a matrix form as follows

$$\begin{aligned} &\frac{d}{ds}\begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix} \\ &= \begin{pmatrix} -\left\langle \uparrow | \dot{\uparrow} \right\rangle & -\exp(i\beta)\left\langle \uparrow | \dot{\downarrow} \right\rangle \\ -\exp(-i\beta)\left\langle \downarrow | \dot{\uparrow} \right\rangle & -\left\langle \downarrow | \dot{\downarrow} \right\rangle \end{pmatrix} \begin{pmatrix} a_\uparrow \\ a_\downarrow \end{pmatrix}, \end{aligned} \tag{C1}$$

where

$$\begin{aligned} \beta(s) &= \int_0^s ds' \left[K_\downarrow(s') - K_\uparrow(s')\right] \\ &= -2\int_0^s ds' \left|\boldsymbol{\kappa}\right|. \end{aligned} \tag{C2}$$

In the adiabatic limit the off-diagonal matrix elements are considered negligibly small, and consequently no transitions between the adiabatic states occur. To calculate the transition probability to lowest order we consider the off diagonal elements as a perturbation [7]. The solution of the unperturbed problem is given by

$$a_\downarrow(s) = a_\downarrow(0)\exp\left(i\gamma_\downarrow\right), \tag{C3}$$

$$a_\uparrow(s) = a_\uparrow(0)\exp\left(i\gamma_\uparrow\right). \tag{C4}$$

Assuming that at some initial point s_0 the system was in the $|\downarrow\rangle$ state, we wish to calculate the probability to find the system in the $|\uparrow\rangle$ state at $s > s_0$. Lowest order correction is obtained by substituting the unperturbed solution in C1

$$\frac{d}{ds}a_\uparrow = -a_\downarrow(0)\exp\left[i\left(\beta+\gamma_\downarrow\right)\right]\left\langle \uparrow | \dot{\downarrow} \right\rangle, \tag{C5}$$

Thus, to lowest order the transition probability is given by

$$p_z = \left| \int_{s_0}^s ds' \exp\left[i\left(\beta+\gamma_\downarrow\right)\right]\left\langle \uparrow | \dot{\downarrow} \right\rangle \right|^2. \tag{C6}$$

Consider the case where $\boldsymbol{\kappa} = \alpha(\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$ is planar with $\varphi = const.$ Using B11

$$\left|\dot{\downarrow}\right\rangle = -\frac{\dot{\theta}}{2}\left|\uparrow\right\rangle, \tag{C7}$$

thus

$$\left\langle \uparrow | \dot{\downarrow} \right\rangle = -\frac{\dot{\theta}}{2}. \tag{C8}$$

Similarly

$$\left\langle \downarrow | \dot{\downarrow} \right\rangle = 0, \tag{C9}$$

thus $\gamma_\downarrow = 0$. Using the above results

$$p_z = \frac{1}{4}\left|\int d\theta \exp(i\zeta)\right|^2, \tag{C10}$$

where

$$\zeta(\theta) = -2\int_0^{s(\theta)} ds' \left|\boldsymbol{\kappa}\right|. \tag{C11}$$

1. The case where $\boldsymbol{\kappa}(s)$ is half circle

Consider the case where $\mathcal{K} = \boldsymbol{\kappa}\cdot\boldsymbol{\sigma}$, where

$$\boldsymbol{\kappa}(s) = \gamma\left(\sqrt{\Lambda^2 - (\gamma s)^2}, 0, \gamma s\right), \tag{C12}$$

γ is a non-negative real constant with dimensionality of 1/length, Λ is a non-negative dimensionless real parameter, and $|s| < \Lambda/\gamma$.

The Zener transition probability is calculated for the case $\Lambda \gtrsim 1$ to lowest order in the adiabatic expansion. The following holds

$$\cos\theta = \frac{\gamma s}{\Lambda}, \tag{C13}$$

and $|\boldsymbol{\kappa}| = \gamma\Lambda$, thus

$$\zeta(\theta) = -2\int_0^{s(\theta)} ds' \left|\boldsymbol{\kappa}\right| = -2\Lambda^2\cos\theta, \tag{C14}$$

FIG. 5: Example of numerical integration of the equation of motion for the case $\Lambda = 5$. On the left the curve $\boldsymbol{\kappa}(s)$ is shown and on the right the evolution of the polarization vector $\mathbf{p}(s)$ on the Bloch sphere is seen.

FIG. 6: Zener probability p_z vs. the parameter Λ calculated numerically.

and

$$p_z = \frac{1}{4}\left|\int\limits_{-\pi}^{0} d\theta \exp\left(-2i\Lambda^2\cos\theta\right)\right|^2 . \tag{C15}$$

Using the identity

$$\int\limits_{0}^{\pi} d\theta \exp\left(iz\cos\theta\right) = \pi J_0\left(z\right), \tag{C16}$$

one finds

$$p_z \simeq \frac{\pi^2}{4} J_0^2\left(2\Lambda^2\right) \qquad (\text{for } \ \Lambda \gtrsim 1). \tag{C17}$$

Fig. 5 shows an example of numerical integration of the equation of motion for the case $\Lambda = 5$. Fig. 5 (a) shows the 'half circle' $\boldsymbol{\kappa}(s)$ curve and Fig. 5 (b) shows the evolution of the polarization vector on the Bloch sphere. Figure 6 shows a numerical calculation of the Zener transition probability p_z as a function of the parameter Λ. As can be seen in Fig. 6, p_z vanishes for a series of points we denote as Λ_n $(n = 1, 2, 3...)$. The first zero of p_z is at $\Lambda_1 = 1.022$. Note however that even though p_z vanishes at the points Λ_n, the evolution becomes truly adiabatic only when $\Lambda >> 1$.

Comparing Eq. C17 with the numerical solution seen in Fig. 6 shows, as expected, good agreement for $\Lambda \gtrsim 1$. For the range $0 \leq \Lambda \lesssim 1$, however, we find that the following can serve as a good approximation

$$p_z \simeq J_0^2\left(\frac{\pi\Lambda^2}{\sqrt{2}}\right) \qquad (\text{for } \ 0 \leq \Lambda \lesssim 1). \tag{C18}$$

2. The case where $\boldsymbol{\kappa}(s)$ is a straight line

We calculate below p_z for the case $\mathcal{K} = \boldsymbol{\kappa}\cdot\boldsymbol{\sigma}$, where $\boldsymbol{\kappa}(s)$ is a straight line

$$\boldsymbol{\kappa}(s) = \Delta\left(0, 1, \gamma s\right), \tag{C19}$$

where Δ and γ are real constants independent of s. For the present case one has

$$\begin{aligned}\zeta(s) &= -2\Delta\int_0^s ds'\sqrt{1+\left(\gamma s'\right)^2} \\ &= -\frac{\Delta}{\gamma}\left[\gamma s\sqrt{1+\left(\gamma s\right)^2} + \sinh^{-1}\left(\gamma s\right)\right]\end{aligned} \tag{C20}$$

and

$$\begin{aligned}&-\frac{1}{2}\int_{\pi}^{0} d\theta \exp\left(i\zeta\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh z}\exp\left[-i\frac{\Delta}{\gamma}\left(\frac{1}{2}\sinh 2z + z\right)\right]dz.\end{aligned} \tag{C21}$$

In the limit $\Delta/\gamma \to \infty$ the phase oscillates rapidly and consequently $p_z \to 0$. The stationary phase points z_n in the complex plane are found from the condition

$$0 = \frac{d}{dz}\left(\frac{1}{2}\sinh 2z + z\right) = \cosh 2z + 1, \tag{C22}$$

thus

$$z_n = i\pi\left(n + \frac{1}{2}\right), \tag{C23}$$

where n integer. Note, however that the term $1/\cosh z$ has poles at the same points. Using the Cauchy's theorem the path of integration can be deformed to pass close to the point $z_{-1} = -i\pi/2$. Since the pole at z_{-1} is a simple one, the principle value of the integral exists. To avoid passing through the pole at z_{-1} a trajectory forming a half circle "above" the pole with radius ε is chosen were $\varepsilon \to 0$. This section gives the dominant contribution which is $i\pi R$, where R is the residue at the pole. Thus one finds:

$$\left| \frac{1}{2} \int_{\pi}^{0} d\theta \exp\left(i\zeta\right) \right| \simeq \exp\left(-\frac{\pi}{2}\frac{\Delta}{\gamma}\right). \tag{C24}$$

The prefactor in front of the exponent is determined by requiring $p_z = 1$ in the limit $\Delta << \gamma$, thus

$$p_z \simeq \exp\left(-\pi\frac{\Delta}{\gamma}\right). \tag{C25}$$

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